Critical Points of Toroidal Belyĭ Maps

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Abstract

A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational function with at most three critical values; we may assume these values are $\{0,\,1,\,\infty\}$. Replacing $\mathbb{P}^1$ with an elliptic curve $E:\ y^2 = x^3 + A\,x + B$, there is a similar definition of a Belyĭ map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Since $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$ is a torus, we call (E, β) a Toroidal Belyĭ pair.

There are many examples of Belyĭ maps $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ associated to elliptic curves; several can be found online at LMFDB. Given such a Toroidal Belyĭ map of degree N, the inverse image $G = \beta^{-1}(\{0,\,1,\,\infty\})$ is a set of N elements which contains the critical points of the Belyĭ map. In this project, we investigate when G is contained in $E(\mathbb{C})_{\text{tors}}$.

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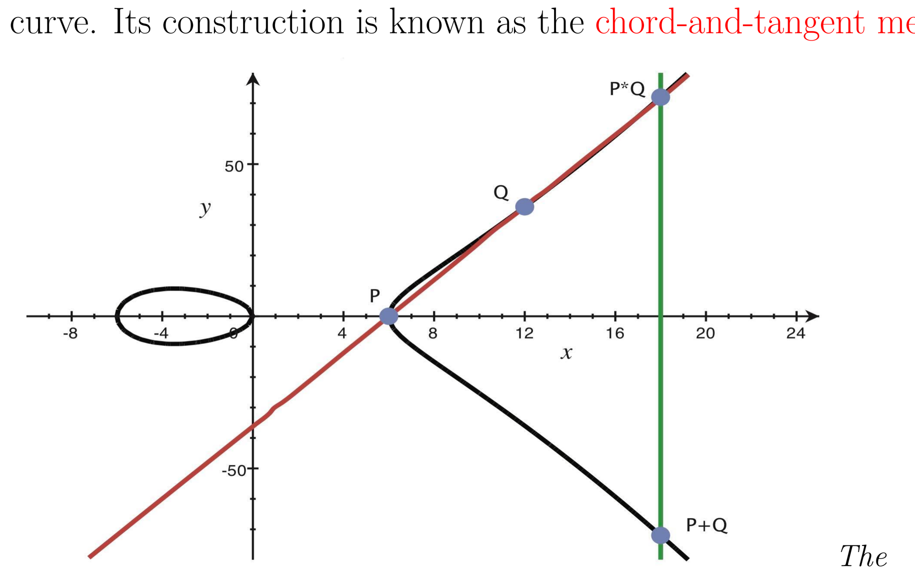
Elliptic Curves

- An elliptic curve, E, is a non-singular curve of genus one. In other words, it is a curve generated by an equation $f(x, y) = 0$ where
$$f(x, y) = y^2 + a_1\,x\,y + a_3\,y - (x^3 + a_2\,x^2 + a_4\,x + a_6)$$
and where all $a_i \in \mathbb{C}$ with O_E being the "point at infinity."
- The set of complex points on an elliptic curve $E(\mathbb{C})$ is a torus.

The Group Law on an Elliptic Curve

- There exists a binary operation $\oplus$ such that $(E(\mathbb{C}), \oplus)$ forms a group with O_E as the identity. This operation is known as the group law on the elliptic curve. Its construction is known as the chord-and-tangent method.



Chord-and-Tangent Construction *The*

- An isogeny is a map $\psi : E \to X$ where E and X elliptic curves such that
$$\psi(P \oplus Q) = \psi(P) \oplus \psi(Q) \text{ for } P, Q \in E(\mathbb{C}).$$

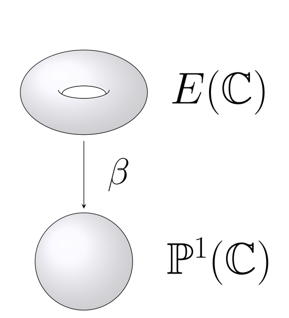


Figure 1: A Toroidal Belyĭ Map

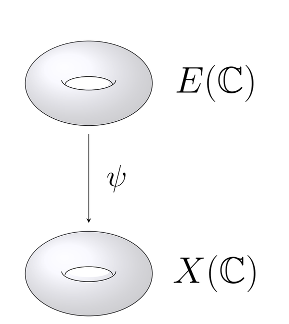


Figure 2: An isogeny

Critical Points and Toroidal Belyĭ Maps

Fix a rational function $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ where $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

- $P \in E(\mathbb{C})$ is a critical point if $\dfrac{\partial f}{\partial x}(P)\,\dfrac{\partial \beta}{\partial y}(P) - \dfrac{\partial f}{\partial y}(P)\,\dfrac{\partial \beta}{\partial x}(P) = 0.$
- $q \in \mathbb{P}^1(\mathbb{C})$ is a critical value if $q = \beta(P)$ for some critical point P.
- $Q \in E(\mathbb{C})$ is a quasi-critical point if $\beta(Q) = \beta(P)$ for critical point P.
- A Belyĭ map is function β as above with ≤ 3 critical values, $\{0, 1, \infty\}$.
- A Toroidal Belyĭ pair is a pair (E, β), where E is an elliptic curve and β is a Belyĭ map associated to E.

Examples of Toroidal Belyĭ Pairs (X, ϕ) with Quasi-Critical Points $\phi^{-1}\big(\{0,\,1,\,\infty\}\big) \subseteq X(\mathbb{C})_{\text{tors}}$

LMFDB Label	Elliptic Curve X	Belyĭ Map $\phi : X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$	Group Generated by $\phi^{-1}\big(\{0,\,1,\,\infty\}\big)$
3T1-3_3_3-a	$y^2 = x^3 + 1$	$\dfrac{1-y}{2}$	Z_3
4T1-4_4_2.2-a	$y^2 = x^3 - x$	x^2	$Z_2 \times Z_2$
4T5-4_4_3.1-a	$y^2 = x^3 + x^2 + 16\,x + 180$	$\dfrac{4\,y + x^2 + 56}{108}$	Z_8
5T4-5_5_3.1.1-a	$y^2 + x\,y = x^3 - 28\,x + 272$	$\dfrac{(x+13)\,y + 3\,x^2 + 4\,x + 220}{432}$	$Z_2 \times Z_{10}$
6T1-6_2.2.2_3.3-a	$y^2 = x^3 + 1$	$-x^3$	$Z_2 \times Z_6$
6T4-3.3_3.3_3.3-a	$y^2 = x^3 - 15\,x + 22$	$\dfrac{8\,(x-2)^2 - (x^2 - 4\,x + 7)\,y}{16\,(x-2)^2}$	Z_6
6T5-6_6_3.1.1.1-a	$y^2 = x^3 + 1$	$\dfrac{(1-y)\,(3+y)}{4}$	$Z_2 \times Z_6$
6T6-6_6_2.2.1.1-a	$y^2 = x^3 + 6\,x - 7$	$\dfrac{(x-1)^3}{27}$	$Z_2 \times Z_4$
6T7-4.2_4.2_3.3-a	$y^2 = x^3 - 10731\,x + 408170$	$\dfrac{11907\,(x-49)}{(x-7)^3}$	$Z_2 \times Z_4$
6T12-5.1_5.1_3.3-b	$y^2 + x\,y + y = x^3 + x^2 - 10\,x - 10$	$27\,\dfrac{(x+4)\,(2\,x^2 - 2\,x - 13) - (x+1)^2\,y}{(x^2 - x - 11)^3}$	$Z_2 \times Z_8$
6T12-5.1_5.1_5.1-a	$y^2 = x^3 + x^2 + 4\,x + 4$	$-16\,\dfrac{(x^2 - 2\,x - 4)\,y + 8\,(x+1)}{(x-4)\,x^5}$	Z_6
8T2-4.4_4.4_2.2.2.2-a	$y^2 = x^3 + x$	$\dfrac{(x+1)^4}{8\,x\,(x^2+1)}$	$Z_2 \times Z_4$
8T7-8_8_2.2.1.1.1.1-a	$y^2 = x^3 - x$	x^4	$Z_2 \times Z_4$

Example #1: 4T1-4_4_2.2-a

Consider the Toroidal Belyĭ pair (E, β) in terms of
$$E : y^2 = x^3 - x \qquad \text{and} \qquad \beta(x, y) = x^2.$$
The quasi-critical points are torsion:

Point	$(0, 0)$	$(1, 0)$	$(-1, 0)$	O_E
Order	2	2	2	1

These points form a group:
$$\beta^{-1}\big(\{0,\,1,\,\infty\}\big) = \big\{(0,0),\,(1,0),\,(-1,0),\,O_E\big\} \simeq Z_2 \times Z_2.$$

Example #2: 4T5-4_4_3.1-a

Consider the Toroidal Belyĭ pair (E, β) in terms of
$$E : y^2 = x^3 + x^2 + 16x + 180 \qquad \text{and} \qquad \beta(x, y) = (4y + x^2 + 56)/108.$$
The quasi-critical points are torsion:

Point	$(4, -18)$	$(22, -108)$	$(-2, 12)$	O_E
Order	4	8	8	1

These points do not form a group.

Example #3: 5T5-5_4.1_4.1-a

Consider the Toroidal Belyĭ pair (E, β) in terms of
$$E : y^2 = x^3 + 5x + 10 \qquad \text{and} \qquad \beta(x, y) = ((x-5)y + 16)/32.$$
The quasi-critical points are not torsion:

Point	$(6, -16)$	$(1, 4)$	$(6, 16)$	$(1, -4)$	O_E
Order	∞	∞	∞	∞	1

These points do not form a group.

Motivating Questions

Given the following:

- (E, β), a Toroidal Belyĭ pair.
- $\Gamma = \beta^{-1}\big(\{0,\,1,\,\infty\}\big)$ as the set of quasi-critical points.

We ask the questions:

1. *When does Γ form a subgroup of $\big(E(\mathbb{C}), \oplus\big)$?*
2. The elements in Γ must be points with finite order whenever Γ is a group. *When are the points in Γ torsion elements in $E(\mathbb{C})$, regardless of Γ being a group?*

Theorem (PRiME 2021)

Given the following:

- (X, ϕ) a Toroidal Belyĭ pair, and $G = \phi^{-1}\big(\{0,\,1,\,\infty\}\big)$ as the set of quasi-critical points.
- $\beta = \phi \circ \psi$, where $\psi : E \to X$ is any non-constant isogeny, and $\Gamma = \beta^{-1}\big(\{0,\,1,\,\infty\}\big)$.

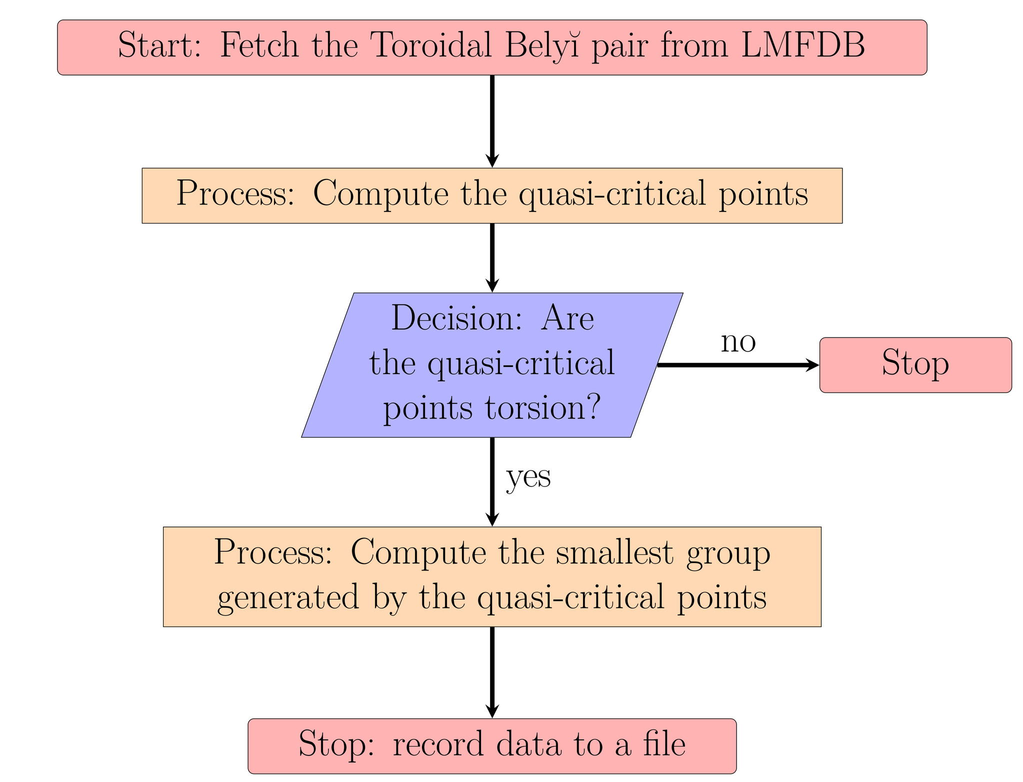
We have the main results:

1. (E, β) is a Toroidal Belyĭ pair.
2. Γ is contained in the torsion in $E(\mathbb{C})$ whenever G is contained in the torsion in $X(\mathbb{C})$.
3. Γ is a group whenever G is a group.

Corollary

There are infinitely many Toroidal Belyĭ pairs where the set of quasi-critical points forms a group.

Computing Examples



Results from Computation

Degree of Belyĭ Map	Total from LMDFB	Total Number of Successfully Processed	Number with Quasi-Critical Points All Torsion
3	1	1 (100%)	1 (100%)
4	2	2 (100%)	2 (100%)
5	7	7 (100%)	1 (14%)
6	35	29 (83%)	7 (24%)
7	73	15 (21%)	0 (0%)
8	94	30 (32%)	2 (7%)
9	39	23 (59%)	0 (0%)
Totals	251	107 (43%)	13 (12%)

Future Work

- Modify the Sage code to run faster in order to get more examples.
- Find more examples of imprimitive Toroidal Belyĭ maps with quasi-critical points that form a group.
- Create an accessible website containing all the information on the data found.

References

[1] PRiME 2021 GitHub Repository.
`https://github.com/PRiME-2021/Algorithms`

[2] Sage Mathematics Software System.
`http://www.sagemath.org`

[3] Ernesto Girondo and Gabino González-Diez. *Introduction to Compact Riemann Surfaces and Dessins d'Enfants.* London Mathematical Society Student Texts, Volume 79 (2012).

[4] The L-functions and Modular Forms Database (LMFDB).
`http://www.lmfdb.org`

[5] Joseph H. Silverman. *The Arithmetic of Elliptic Curves.* Graduate Texts in Mathematics, Volume 106. Second Edition (2009).

[6] Joseph H. Silverman and John T. Tate. *Rational points on elliptic curves.* Undergraduate Texts in Mathematics (1993).

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